Solution 3 by Moti Levy, Rehovot, Israel. Let $a=\cosh^2 t$ then our limit becomes

$$L=\lim_{x\to\infty} x^a\left(\left(\Gamma(x+1)^{\frac{1}{x}}\right)^{1-a}-\left(\Gamma(x+2)^{\frac{1}{x+1}}\right)^{1-a}\right)$$
$$=\lim_{x\to\infty} x^a\left(\Gamma(x+1)^{\frac{1}{x}}\right)^{1-a}\left(1-\left(\frac{\Gamma(x+2)^{\frac{1}{x+1}}}{\Gamma(x+1)^{\frac{1}{x}}}\right)^{1-a}\right).$$

Using the asymptotic expression for $\Gamma(x+1)$,

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x,$$

we obtain

$$\Gamma(x+1)^{\frac{1}{x}}\sim\frac{x}{e},$$
$$\Gamma(x+2)^{\frac{1}{x+1}}\sim\frac{x+1}{e}.$$

Hence

$$L=\lim_{x\to\infty} x^a\left(\frac{x}{e}\right)^{1-a}\left(1-\left(\frac{x+1}{x}\right)^{1-a}\right)=e^{a-1}\lim_{x\to\infty} x\left(1-\left(\frac{x+1}{x}\right)^{1-a}\right)$$
$$=e^{a-1}\lim_{x\to\infty}\frac{1-\left(\frac{x+1}{x}\right)^{1-a}}{x^{-1}}.$$

Applying L'Hopital's rule

$$L=e^{a-1}\lim_{x\to\infty}\frac{-(1-a)\left(\frac{x+1}{x}\right)^{-a}\left(-x^{-2}\right)}{\left(-x^{-2}\right)}=(a-1)\,e^{a-1}.$$

$$\lim_{x\to\infty} x^{\cosh^2 t}\left(\left(\Gamma(x+1)^{\frac{1}{x}}\right)^{-\sinh^2 t}-\left(\Gamma(x+2)^{\frac{1}{x+1}}\right)^{-\sinh^2 t}\right)$$
$$=(\cosh$$

$^2t-1e^{\left(\cosh^2 t-1\right)}=e^{\sinh^2 t}\sinh^2 t.$

Also solved by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy; Moubinool Omarjee, Paris, France and the proposers.

149. *Proposed by Arkady Alt, San Jose, California, USA.* Let D be the set of strictly decreasing sequences of positive real numbers with first term equal to 1. For given positive p, r and any $x_{\mathbb{N}}=(x_1,x_2,\ldots)\in D$, let $S(x_{\mathbb{N}})=\sum_{n=1}^{\infty}\frac{x_n^{p+r}}{x_{n+1}^p}$ if this series converges and define $S(x_{\mathbb{N}})=\infty$ otherwise. Find $\inf\{S(x_{\mathbb{N}})|x_{\mathbb{N}}\in D\}$.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

The answer is $A(p,r)\overset{\text{def}}{=}\left(\frac{(p+r)^{p+r}}{r^rp^p}\right)^{1/r}$.

Consider a sequence $x_{(\mathbb{N})} \in D$. For $n > 2$ we have, using Hölder's inequality:

$$\begin{aligned}\sum_{k=1}^{n-1} x_k^r &= \sum_{k=1}^{n-1} \frac{x_k^r}{x_{k+1}^{pr/(p+r)}} \cdot x_{k+1}^{pr/(p+r)} \\ &\leq \left(\sum_{k=1}^{n-1}\left(\frac{x_k^r}{x_{k+1}^{pr/(p+r)}}\right)^{\frac{p+r}{r}}\right)^{\frac{r}{p+r}} \left(\sum_{k=1}^{n-1}\left(x_{k+1}^{pr/(p+r)}\right)^{\frac{p+r}{p}}\right)^{\frac{p}{p+r}} \\ &\leq \left(\sum_{k=1}^{n-1} \frac{x_k^{p+r}}{x_{k+1}^p}\right)^{\frac{r}{p+r}} \left(\sum_{k=1}^{n-1} x_{k+1}^r\right)^{\frac{p}{p+r}}\end{aligned}$$

So we have proved that

$$\left(\sum_{k=1}^{n-1} x_k^r\right)^{1+p/r} \leq \left(\sum_{k=1}^{n-1} \frac{x_k^{p+r}}{x_{k+1}^p}\right)\left(\sum_{k=2}^{n} x_k^r\right)^{p/r} \tag{1}$$

On the other hand, using the arithmetic mean-geometric mean inequality, we have for $x, t > 0$ that

$$\frac{1+x}{1+t} = \frac{1+t(x/t)}{1+t} \geq \left(\frac{x}{t}\right)^{t/(1+t)}$$

Applying this with $x = \sum_{k=2}^{n-1} x_k^r$ and $t = p/r$ we see that

$$\left(\sum_{k=1}^{n-1} x_k^r\right)^{1+p/r} \geq \frac{(1+t)^{1+t}}{t^t}\left(\sum_{k=2}^{n-1} x_k^r\right)^{p/r}$$

Or equivalently

$$\left(\sum_{k=1}^{n-1} x_k^r\right)^{1+p/r} \geq A(p,r)\left(-x_n^r + \sum_{k=2}^{n} x_k^r\right)^{p/r} \tag{2}$$

Combining (1) and (2) we get

$$\sum_{k=1}^{n-1} \frac{x_k^{p+r}}{x_{k+1}^p} \geq A(p,r)\left(1 - \frac{x_n^r}{\sum_{k=2}^{n} x_k^r}\right)^{p/r} \tag{3}$$

and this is also valid for $n = 2$. Now, let us consider two cases:

- If $\sum_{k=1}^{\infty} x_k^r = +\infty$ then from the inequality

$$0 \leq \frac{x_n^r}{\sum_{k=2}^{n} x_k^r} \leq \frac{x_1^r}{\sum_{k=2}^{n} x_k^r}$$

  we conclude that

$$\lim_{n\to\infty} \frac{x_n^r}{\sum_{k=2}^{n} x_k^r} = 0$$

- If $\sum_{k=1}^{\infty} x_k^r = \ell < +\infty$, then clearly $\lim_{n\to\infty} x_n^r = 0$ and again

$$\lim_{n\to\infty} \frac{x_n^r}{\sum_{k=2}^{n} x_k^r} = 0$$

Combining the above results and letting n tend to infinity in (3) we conclude that $S(x_{\mathbb{N}}) \geq A(p,r)$, and consequently

$$\inf\{S(x_{\mathbb{N}})|x_{\mathbb{N}} \in D\} \geq A(p,r) \tag{4}$$

Conversely, consider the sequence $a_{\mathbb{N}} = (a_n)_{n\geq 1}$ defined by $a_n = \alpha^{n-1}$ with $\alpha = \left(\frac{p}{p+r}\right)^{1/r} < 1$. Clearly we have

$$S(a_{\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{\alpha^{(r+p)(n-1)}}{\alpha^{pn}} = \frac{1}{\alpha^p} \cdot \frac{1}{1-\alpha^r} = \left(\frac{p+r}{p}\right)^{p/r} \cdot \frac{p+r}{r} = A(p,r).$$

Hence,$\inf\{S(x_{\mathbb{N}})|x_{\mathbb{N}} \in D\} = A(p,r)$ and the lower bound is in fact attained on a geometric sequence.

Also solved by the proposer.

150. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Find

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+n}\frac{H_{k+n}^3}{k+n},$$

where $H_n = \sum_{j=1}^{n} 1/j$ denotes the nth harmonic number.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. We will use a general principle. Consider a an analytic f in the unit disk $D(0,1)$, and suppose that its power series expansion is given by $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Now, using the integral form of the remainder we may write for $|z| < 1$ the following

$$\begin{aligned}\sum_{n=1}^{\infty} a_{n+k} z^{n+k} &= f(z) - \sum_{n=1}^{k} \frac{f^{(n)}(0)}{n!} z^n \\ &= \frac{z^{k+1}}{k!}\int_0^1 (1-t)^k f^{(k+1)}(tz)dt\end{aligned}$$

It follows that for $|w| < 1$ we have

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{n+k} z^{n+k}\right) w^k &= z\int_0^1 \left(\sum_{k=1}^{\infty} \frac{(f')^{(k)}(tz)}{k!}(zw(1-t))^k\right) dt \\ &= z\int_0^1 \big(f'(tz + zw(1-t)) - f'(tz)\big)dt \\ &= \left[\frac{1}{1-w} f(zw + tz(1-w)) - f(tz)\right]_{t=0}^{t=1} \\ &= \frac{f(z) - f(zw)}{1-w} - f(z) \qquad (1)\end{aligned}$$

Now in our case we have $a_n = H_n^3/n$ and $f(z) = \sum_{n=1}^{\infty} \frac{H_n^3}{n} z^n$. Since the series defining $f(-1)$ does converge by the alternating series test (this is not straightforward but it can be proved that the coefficients decrease to 0 starting from a certain index), it is easy to show that uniformly in $z \in (-1,0)$ we have $\sum_{n=1}^{\infty} a_{n+k} z^{n+k} = \mathcal{O}(\log^3 k)$